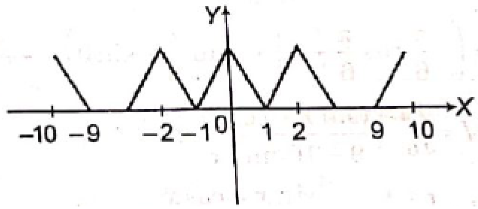
51.(4) Given $f(x)=\begin{cases} x-[x], & \text{if } [x] \text{ is odd.} \\ 1+[x]-x, & \text{if } [x] \text{ is even.}\end{cases}$

$f(x)$ and $\cos\pi x$ both are periodic with period 2 and both are even.

$$\therefore \quad \int_{-10}^{10} f(x)\cos\pi x\,dx = 2\int_{0}^{10} f(x)\cos\pi x\,dx$$

$$=10\int_0^2 f(x)\cos\pi x\,dx$$



Now, $\int_0^1 f(x)\cos\pi x\,dx$

$$=\int_0^1 (1-x)\cos\pi x\,dx = -\int_0^1 u\cos\pi u\,du \quad \text{and} \quad \int_1^2 f(x)\cos\pi x\,dx = \int_1^2 (x-1)\cos\pi x\,dx = -\int_0^1 u\cos\pi u\,du$$

$$\therefore \quad \int_{-10}^{10} f(x)\cos\pi x\,dx = -20\int_0^1 u\cos\pi u\,du = \frac{40}{\pi^2} \quad \Rightarrow \quad \frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos\pi x\,dx = 4$$

52.(B) Given, $F(x)=\int_0^{x^2} f(\sqrt{t})dt$

$\therefore \quad F'(x)=2x\,f(x)$

Also, $F'(x)=f'(x)$

$\Rightarrow \quad 2xf(x)=f'(x) \Rightarrow \dfrac{f'(x)}{f(x)}=2x \quad \Rightarrow \quad \int \dfrac{f'(x)}{f(x)}dx = \int 2x\,dx$

$\Rightarrow \quad \ln f(x)=x^2+c \Rightarrow f(x)=e^{x^2+c} \quad \Rightarrow \quad f(x)=K\,e^{x^2} \qquad [K=e^0\ ]$

Now, $f(0)=1$

$\therefore \quad 1=K$ Hence, $f(x)=e^{x^2}$; $F(2)=\int_0^4 e^t dt = [e^t]_0^4 = e^4-1$

53.(C) Given $\int_0^x \sqrt{1-\{f'(t)\}^2}\,dt = \int_0^x f(t)dt,\ 0\le x\le 1$

Differentiating both sides w.r.t. x by using Leibnitz's rule, we get

$\sqrt{1-\{f'(x)\}^2} = f(x) \Rightarrow f'(x)=\pm\sqrt{1-\{f(x)\}^2} \quad \Rightarrow \quad \int \dfrac{f'(x)}{\sqrt{1-\{f(x)\}^2}}dx = \pm\int dx \Rightarrow \sin^{-1}\{f(x)\}=\pm x + c$

Put $x=0 \Rightarrow \sin^{-1}\{f(0)\}=c \quad \Rightarrow \quad c=0 \quad \Rightarrow \quad f(x)=\pm\sin x$

But $f(x)\ge 0,\ \forall\ x\in[0,1]$

$\therefore \quad f(x)=\sin x$

As we know that, $\sin x < x,\ \forall\ x>0$

$\therefore \quad \sin\left(\dfrac{1}{2}\right) < \dfrac{1}{2}$ and $\sin\left(\dfrac{1}{3}\right) < \dfrac{1}{3}$

$\Rightarrow \quad f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

54.(A) Given, $\int_0^x f(t)\,dt = x + \int_x^1 t\, f(t)dt$

On differentiating both sides w.r.t x, we get

$f(x) = 1 - x\, f(x).1 \Rightarrow (1+x)f(x) = 1 \Rightarrow \quad f(x) = \frac{1}{1+x} \Rightarrow f(1) = \frac{1}{1+1} = \frac{1}{2}$

55.(A) $\lim_{x\to 1}\int_4^{f(x)} \frac{2t}{x-1}dt = \lim_{x\to 1}\frac{\int_4^{f(x)} 2t\,dt}{x-1}$ [using L'Hospital's rule]

$= \lim_{x\to 1}\frac{2f(x).f'(x)}{1} = 2f(1).f'(1) = 8f'(1)$ $[\because f(1) = 4]$

56.(C) Here, $f(x) + 2x = (1-x)^2.\sin^2 x + x^2 + 2x$ ....(i)

Where, $P : f(x) + 2x = 2(1+x)^2$ ....(ii)

$\therefore \quad 2(1+x^2) = (1-x)^2\sin^2 x + x^2 + 2x$

$\Rightarrow \quad (1-x)^2\sin^2 x = x^2 - 2x + 2$

$\Rightarrow \quad (1-x)^2\sin^2 x = (1-x)^2 + 1 \Rightarrow (1-x)^2\cos^2 x = -1$

Which is never possible.

$\therefore$ P is false.

Again $let Q : h(x) = 2f(x) + 1 - 2x(1+x)$

Where, $h(0) = 2f(0) + 1 - 0 = 1$

$h(1) = 2f(1) + 1 - 4 = -1$, as $h(0)h(1) < 0$

$\Rightarrow \quad h(x)$ must have a solution.

$\therefore$ Q is true.

57.(B) Here, $f(x) = (1-x)^2.\sin^2 x + x^2 \geq 0, \forall x$

And $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \log t\right) f(t)\,dt$

$\Rightarrow \quad g'(x) = \left\{\frac{2(x-1)}{(x+1)} - \log x\right\}.\underset{+ve}{f(x)}$ .... (i)

For $g'(x)$ to be increasing or decreasing,

let $\phi(x) = \frac{2(x-1)}{(x+1)} - \log x$

$\phi'(x) = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-(x-1)^2}{x(x+1)^2}$

$\phi'(x) < 0$, for $x > 1 \Rightarrow \phi(x) < \phi(1) \Rightarrow \phi(x) < 0$ .... (ii)

From equations (i) and (ii), we get

$g'(x) < 0$ for $x \in (1, \infty)$

$\therefore \quad g(x)$ is decreasing for $x \in (1, \infty)$

58. Given, $f(x)=\begin{vmatrix} \sec x & \cos x & \operatorname{cosec}x.\cot x+\sec^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$

Applying $R_3 \to \frac{1}{\cos x}R_3$,

$$f(x)=\cos x\begin{vmatrix} \sec x & \cos x & \operatorname{cosec}x.\cot x+\sec^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ \sec x & \cos x & \cos x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3 \Rightarrow f(x)$

$$=\cos x\begin{vmatrix} 0 & 0 & \operatorname{cosec}x.\cot x+\sec^2 x-\cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ \sec x & \cos x & \cos x \end{vmatrix}$$

$$=(\operatorname{cosec}x.\cot x+\sec^2 x-\cos x).(\cos^3 x-\cos x).\cos x$$

$$=-\left[\frac{\sin^2 x+\cos^3 x-\cos^3 x.\sin^2 x}{\sin^2 x.\cos^2 x}\right].\cos^2 x.\sin^2 x$$

$$=-\sin^2 x-\cos^3 x(1-\sin^2 x)=-\sin^2 x-\cos^5 x$$

$\therefore \quad \int_0^{\pi/2} f(x)dx = -\int_0^{\pi/2}(\sin^2 x+\cos^5 x)dx \quad \left[\because \int_0^{\pi/2}\sin^m x\cdot\cos^n x\,dx = \dfrac{\sqrt{\frac{m+1}{2}}\sqrt{\frac{n+2}{2}}}{2\sqrt{\frac{m+n+2}{2}}}\right]$

$$\int_0^{\pi/2} f(x)dx = -\left\{\frac{\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{1}{2}}}{2\sqrt{2}}+\frac{\sqrt{\frac{6}{2}}\cdot\sqrt{\frac{1}{2}}}{2\sqrt{\frac{7}{2}}}\right\} = -\left\{\frac{1/2\cdot\pi}{2}+\frac{2\sqrt{\pi}}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}\right\} = -\left\{\frac{\pi}{4}+\frac{8}{16}\right\} = -\left(\frac{15\pi+32}{60}\right)$$

59. $f(x)=\int_1^x \frac{\ln t}{1+t}dt$ for $x>0$

Now, $f(1/x)=\int_1^{1/x}\frac{\ln t}{1+t}dt$

Put $t=1/u \quad \Rightarrow \quad dt=(-1/u^2)du$

$$f(1/x)=\int_1^x \frac{\ln(1/u)}{1+1/u}\cdot\frac{(-1)}{u^2}du = \int_1^x \frac{\ln u}{u(u+1)}du = \int_1^x \frac{\ln t}{t(1+t)}dt$$

Now, $f(x)+f\left(\frac{1}{x}\right)=\int_1^x \frac{\ln t}{(1+t)}dt+\int_1^x \frac{\ln t}{t(1+t)}dt = \int_1^x \frac{(1+t)\ln t}{t(1+t)}dt = \int_1^x \frac{\ln t}{t}dt = \frac{1}{2}[(\ln t)^2]_1^x = \frac{1}{2}(\ln x)^2$

Put, $x=e$; $f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}(\ln e)^2=\frac{1}{2}$

60. Let $t=b-a$ and $a+b=4 \quad \Rightarrow \quad t=4-a-a \quad \Rightarrow \quad t=4-2a \quad \Rightarrow \quad a=2-\frac{t}{2}$

and $t=b-(4-b) \quad \Rightarrow \quad t=2b-4 \quad \Rightarrow \quad \frac{t}{2}=b-2 \quad \Rightarrow \quad b=2+\frac{t}{2}$

Again, $a < 2$

$\Rightarrow \quad 2-\dfrac{t}{2} < 2 \quad \Rightarrow \quad \dfrac{t}{2} > 0 \quad \Rightarrow \quad t > 0$

Now, $\displaystyle\int_0^a g(x)\,dx + \int_0^b g(x)\,dx = \int_0^{2-t/2} g(x)\,dx + \int_0^{2+t/2} g(x)\,dx$

Let $\displaystyle F(t) = \int_0^{2-t/2} g(x)\,dx + \int_0^{2+t/2} g(x)\,dx$

For $t > 0$, $F'(t) = -\dfrac{1}{2}g\left(2-\dfrac{t}{2}\right)+\dfrac{1}{2}g\left(2+\dfrac{t}{2}\right)$ [Using Leibnitz's rule]

$$= \frac{1}{2}g\left(2+\frac{t}{2}\right)-\frac{1}{2}g\left(2-\frac{t}{2}\right)$$

Again, $\dfrac{dg}{dx} > 0, \forall x \in R$

Now, $2-t/2 < 2+t/2$

$\therefore \quad t > 0$

We get $g(2+t/2)-g(2-t/2) > 0, \forall t > 0$

So, $F'(t) > 0, \forall\, t > 0$

Hence, $F(t)$ increases with t, therefore $F(t)$ increases as $(b-a)$ increases.